Ratio Derivatives

ANTHONY QUAS, *Gordano Comprehensive School, Bristol*

The author had this idea and wrote the article when he was a sixth former at Gordano Comprehensive School, Bristol. He is now reading mathematics at King's College, Cambridge.

Many common functions are differentiable, i.e. the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is defined. This definition uses subtraction and division. We define here another limit $f^*(x)$ using instead division and powers, i.e.

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h}.$$

We shall write $\rho_x(f(x))$ for $f^*(x)$, analogous to $(\mathrm{d}/\mathrm{d}x)(f(x))$. Then, if λ is a constant, it is easy to see that

$$\rho_x(\lambda f(x)) = \rho_x(f(x)).$$

This corresponds to

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)+\lambda) = \frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

The formula for differentiating a product is actually easier for this 'ratio derivative' than for the usual derivative. For

$$\rho_x(f(x)g(x)) = \lim_{h\to 0}\left(\frac{f(x+h)g(x+h)}{f(x)g(x)}\right)^{1/h} = \rho_x(f(x))\rho_x(g(x)).$$

As an example,

$$\rho_x(\mathrm{e}^x) = \lim_{h\to 0}\left(\frac{\mathrm{e}^{x+h}}{\mathrm{e}^x}\right)^{1/h} = \lim_{h\to 0}(\mathrm{e}^h)^{1/h} = \mathrm{e}.$$

The well-known limit

$$\lim_{h\to 0}(1+h)^{1/h} = \mathrm{e}$$

enables us to evaluate various ratio derivatives. For example,

$$\rho_x(x) = \lim_{h\to 0}\left(\frac{x+h}{x}\right)^{1/h} = \lim_{h\to 0}\left[\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x} = \mathrm{e}^{1/x},$$

whence, from the product formula,

$$\rho_x(x^n) = e^{n/x},$$

when n is a positive integer.

The connection between the ratio derivative and the usual derivative is given by

$$\begin{aligned}\ln f^*(x) &= \ln \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h} \\ &= \lim_{h\to 0} \ln\left(\frac{f(x+h)}{f(x)}\right)^{1/h} \\ &= \lim_{h\to 0} \frac{\ln f(x+h) - \ln f(x)}{h} \\ &= (\ln f(x))' \\ &= \frac{f'(x)}{f(x)},\end{aligned}$$

so that

$$f'(x) = f(x)\ln f^*(x), \qquad f^*(x) = e^{f'(x)/f(x)}.$$

Thus, for example, we can recover the formula for $(d/dx)(x^n)$:

$$\frac{d}{dx}x^n = x^n \ln \rho_x(x^n) = x^n \frac{n}{x} = nx^{n-1},$$

and the formula for differentiating a product follows from the ratio derivative of a product:

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f(x)g(x)\ln(f^*(x)g^*(x)) \\ &= f(x)g(x)\ln f^*(x) + f(x)g(x)\ln g^*(x) \\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

We can define the analogue of integration by writing

$$P_x(f^*(x)) = f(x).$$

The connection between P_x and the usual integration comes from

$$\begin{aligned} & \ln f^*(x) = \frac{f'(x)}{f(x)} \\ \Rightarrow \quad & \int \ln f^*(x)\, dx = \int \frac{f'(x)}{f(x)}\, dx = \ln f(x) \\ \Rightarrow \quad & f(x) = \exp\left(\int \ln f^*(x)\, dx\right).\end{aligned}$$

Thus

$$P_x(u(x)) = \exp\left(\int \ln u(x)\ \mathrm{d}x\right).$$

We note that, since the ordinary indefinite integral involves an arbitrary additive constant, the indefinite ratio integral has an arbitrary multiplicative constant.

It is reasonable to define the definite ratio integral by

$${}_a^bP_x(u(x)) = \frac{[P_x(u(x))]_b}{[P_x(u(x))]_a},$$

so that

$${}_a^bP_x(u(x)) = \frac{\left[\exp\left(\int \ln u(x)\ \mathrm{d}x\right)\right]_b}{\left[\exp\left(\int \ln u(x)\ \mathrm{d}x\right)\right]_a} = \exp\left(\left[\int \ln u(x)\ \mathrm{d}x\right]_a^b\right),$$

i.e.

$${}_a^bP_x(u(x)) = \exp\left(\int_a^b \ln u(x)\ \mathrm{d}x\right).$$

This can be used to obtain an approximation for $n!$. We divide the interval from a to b into equal intervals of width h. Then, for small h,

$$\int_a^b \ln f(x)\ \mathrm{d}x \simeq h[\ln f(a) + \ln f(a+h) + \ln f(a+2h) + \ldots + \ln f(b-h)]$$

$$= \ln[f(a)f(a+h)f(a+2h)\ldots f(b-h)]^h.$$

Thus

$${}_a^bP_x(f(x)) \simeq [f(a)f(a+h)f(a+2h)\ldots f(b-h)]^h.$$

Now put $f(x) = x$, $a = 1/n$, $b = 1$ and $h = 1/n$. Then

$$\begin{aligned} {}_{1/n}^{\ 1}P_x(x) &= \exp\left(\int_{1/n}^1 \ln x\ \mathrm{d}x\right) \\ &= \exp([x\ln x - x]_{1/n}^1) \\ &= \exp\left(-1 - \frac{1}{n}\ln\frac{1}{n} + \frac{1}{n}\right) \\ &= \exp\left(-1 + \frac{1}{n}\right)n^{1/n}, \end{aligned}$$

so that, for large n,

$$\exp\left(-1+\frac{1}{n}\right)n^{1/n} \simeq \left(\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\times\ldots\times\frac{n-1}{n}\right)^{1/n}.$$

Thus

$$e^{-n+1}n \simeq \frac{(n-1)!}{n^{n-1}},$$

or

$$(n-1)! \simeq \frac{n^n}{e^{n-1}},$$

or

$$n! \simeq \frac{(n+1)^{n+1}}{e^n}.$$

If we compare values, this approximation seems unimpressive. For example, $5! \simeq 314$ (in fact $5! = 120$) and $10! \simeq 12\,950\,000$ (instead of $3\,528\,800$). However, if the graphs of the logarithms of $n!$ and the approximation are plotted, a very close correspondence is immediately noticeable. The two graphs have almost exactly the same form. And our approximation bears a certain resemblance to a famous approximation to $n!$, namely *Stirling's formula*, which is

$$n! \simeq \sqrt{2\pi}\,\frac{n^{n+\frac{1}{2}}}{e^n}.$$

The usual derivative has a geometrical interpretation in terms of the slope of the graph and a physical interpretation in terms of rates of change (for example, ds/dt gives speed, where s is distance and t is time). It would be interesting to know if there are corresponding geometrical and physical interpretations of the ratio derivative. Perhaps readers could help.

We leave readers with an exercise. Produce a formula for the ratio derivative of the sum of two functions.

The biggest prime in the world

Joseph Mclean, now Research Assistant in the Department of Computing Science at the University of Strathclyde, writes:

'It has come to my attention, and my sources are many and reliable, that the new largest known prime is the Mersenne prime

$$2^{216091} - 1$$

with an exponent roughly double that of the previous largest (see Volume 19 No. 2, page 46).'